

ON UNIFORM ESTIMATES FOR POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We consider the semilinear elliptic equation $\Delta u = W'(u)$ with Dirichlet boundary conditions in a Lipschitz, possibly unbounded, domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions on the potential W , we deduce a condition on the size of the domain that implies the existence of a positive solution satisfying a uniform pointwise estimate. Here, uniform means that the estimate is independent of Ω . The main advantage of our approach is that it allows us to remove a restrictive monotonicity assumption on W that was imposed in the recent paper by G. Fusco, F. Leonetti and C. Pignotti [50]. In addition, we can remove a non-degeneracy condition on the global minimum of W that was assumed in the latter reference. Our approach is based on a refinement of a useful result of P. Clément and G. Sweers [32], concerning the behavior of global minimizers of the associated energy over large balls, subject to Dirichlet conditions. As an application of our main result, we can generalize a result of P. Hess [54] and D. G. De Figueiredo [42], concerning semilinear elliptic nonlinear eigenvalue problems. Moreover, we study the length of the boundary layer of global minimizers of the corresponding singular perturbation problem.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

A problem that has received considerable attention in the literature is the study of the structure of solutions $(\lambda, u) \in \mathbb{R} \times C^{2,\alpha}(\bar{\mathcal{D}})$, depending on the nonlinearity f , of the semilinear elliptic nonlinear eigenvalue problem

$$\Delta u + \lambda f(u) = 0, \quad x \in \mathcal{D}; \quad u(x) = 0, \quad x \in \partial\mathcal{D}, \quad (1.1)$$

where \mathcal{D} is typically a smooth bounded domain. To this end, the main approaches used include the method of upper and lower solutions, bifurcation techniques, topological and variational methods (see [62], [76], [78] and the references therein).

Recently, G. Fusco, F. Leonetti and C. Pignotti considered in [50] the semilinear elliptic problem

$$\begin{cases} \Delta u = W'(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.2)$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with nonempty Lipschitz boundary (see for instance [46]), under the following assumptions on the C^2 function $W : \mathbb{R} \rightarrow \mathbb{R}$, which we will often refer to as a potential:

(a): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t), \quad t \in [0, \infty), \quad t \neq \mu,$$

$$W(-t) \geq W(t), \quad t \in [0, \infty);$$

(b): $W'(t) \leq 0$, $t \in (0, \mu)$;

(c): $W''(\mu) > 0$.

For a typical example of such a potential, see (1.13) below. We stress that, in the case where the domain is unbounded, the boundary conditions in (1.2) *do not* refer to $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$ with $x \in \Omega$. Note that (1.1) can be related to (1.2) via a simple rescaling (see the relation between (3.9) and (3.10) below).

For $x \in \mathbb{R}^n$, $\rho > 0$, we let

$$B_\rho(x) = \{y \in \mathbb{R}^n : |y - x| < \rho\}, \quad B_\rho = B_\rho(0),$$

$$A + B = \{x + y : x \in A, y \in B\}, \quad A, B \subset \mathbb{R}^n,$$

and denote by $d(x, E)$ the Euclidean distance of the point $x \in \mathbb{R}^n$ from the set $E \subset \mathbb{R}^n$, and by $|E|$, unless specified otherwise, the n -dimensional Lebesgue measure of E (see [46]). By $\mathcal{O}(\cdot)$, $o(\cdot)$ we will denote the standard Landau's symbols.

The main result of [50] was the following:

Theorem 1.1. Assume Ω and W as above. There are positive constants R^* , $r^* \in (0, R^*)$, $a^* \in (0, \mu)$, k, K , depending only on W and n , such that if Ω contains a closed ball of radius R^* , then problem (1.2) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying

$$0 < u(x) < \mu, \quad x \in \Omega, \quad (1.3)$$

$$\mu - a^* < u(x), \quad x \in \Omega_{R^*} + B_{r^*}, \quad (1.4)$$

and

$$\mu - u(x) \leq K e^{-kd(x, \partial\Omega)}, \quad x \in \Omega, \quad (1.5)$$

where

$$\Omega_{R^*} = \{x \in \Omega : d(x, \partial\Omega) > R^*\}. \quad (1.6)$$

The approach of [50] to the proof of Theorem 1.1 is variational, involving the construction of various judicious radial comparison functions, see also [5]. In our opinion, their argument boils down to the construction of a lower solution to (1.2), see [15], whose building blocks, after a translation, are radial solutions of

$$\Delta u + a^2(\mu - u) = 0 \text{ in } B_R \text{ (with } a^2 < W''(\mu)), \quad -\Delta u = 0 \text{ in } \Omega \setminus B_R,$$

and the constant function 0, for large R (note that assumption (b) implies that solutions of (1.2) are supharmonic). We note that, once (1.4) is established, the proof of the exponential decay estimate (1.5), given in [50], can be simplified considerably by employing Lemma 4.2 in [47], making use of the non-degeneracy condition (c) (the constants in Theorem 1.1 can be chosen so that $W''(t) > 0$, $t \in [\mu - a^*, \mu]$). Moreover, an examination of the proof of Lemma 2.1 in [50] (see Lemma A.1 herein) shows that assumption (a) above can be relaxed to

(a'): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t), \quad t \in [0, \mu), \quad W(t) \geq 0, \quad t \geq \mu,$$

$$W(-t) \geq W(t), \quad t \in [0, \mu].$$

The main purpose of this note is to show that relation (1.4) can be established in a simple manner *without assuming* the monotonicity condition (b), and in fact we will prove a stronger version of it. We will accomplish this, loosely speaking, by using translations of a solution of

$$\Delta u = W'(u), \quad x \in B_R; \quad u(x) = 0, \quad x \in \partial B_R,$$

which minimizes the associated energy, as a lower solution after we have extended it by zero outside of B_R . In passing, we remark that a similar monotonicity assumption to **(b)** also appears in [5], in the context of variational systems of the form (1.2), where $W : \mathbb{R}^n \rightarrow \mathbb{R}$, with the obvious notation (see also Remark 1.4 and Lemma A.3 below). Moreover, we remove completely the non-degeneracy condition **(c)** from the proof of (1.4).

Our main result is

Theorem 1.2. Assume that Ω is as above, and that $W \in C^2$ satisfies **(a')**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is determined from the relation

$$\mathbf{U}(D') = \mu - \epsilon, \quad (1.7)$$

where in turn \mathbf{U} is the only function in $C^2[0, \infty)$ that satisfies

$$\mathbf{U}'' = W'(\mathbf{U}), \quad s > 0; \quad \mathbf{U}(0) = 0, \quad \lim_{s \rightarrow \infty} \mathbf{U}(s) = \mu, \quad (1.8)$$

(see Remark 1.1 below). There exists an $R' > D$, depending only on ϵ , D , W , and n , such that if Ω contains some closed ball of radius R' then problem (1.2) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying (1.3), and

$$\mu - \epsilon \leq u(x), \quad x \in \Omega_{R'} + B_{(R'-D)}, \quad (1.9)$$

where $\Omega_{R'}$ was previously defined in (1.6). Moreover, if $W''(\mu) > 0$ then estimate (1.5) holds true.

As will be apparent from the proof, see in particular the comments leading to Proposition 4.1 below, the delicacy of our result is that the constant D' is independent of n .

Remark 1.1. The existence and uniqueness of such a solution \mathbf{U} of the ordinary differential equation $u'' = W'(u)$ follows readily from **(a')** by phase plane analysis, using the fact that the latter equation has the conserved quantity $e(s) = \frac{1}{2}(u')^2 - W(u)$, see for instance Chapter 2 in [13] or page 135 in [82] (for a more analytic approach, we refer to [16]). We note that

$$\mathbf{U}'(s) > 0, \quad s \geq 0. \quad (1.10)$$

Remark 1.2. Similar assertions hold for the Robin boundary value problem:

$$\Delta u = W'(u), \quad x \in \Omega; \quad \frac{\partial u}{\partial \nu} + b(x)u = 0, \quad x \in \partial\Omega,$$

where ν denotes the outward unit normal vector to the boundary of Ω , assuming here that the latter is at least C^1 , with $b \in C^{1+\alpha}(\partial\Omega)$, $\alpha > 0$, being nonnegative (so that the constant μ is a positive upper solution, see [74]). Moreover, as in [50], we can possibly study some problems with mixed boundary conditions.

Remark 1.3. A sufficient, and easy to check, condition for the uniqueness of a positive solution of (1.2), in *any smooth bounded domain*, is $W'(t)/t$ being strictly increasing in $(0, \infty)$ (see [23]). Related conditions can also be found in [80]. Another sufficient condition, which on the other hand depends on the smooth bounded domain Ω , is

$$W'(t_2) - W'(t_1) \geq \lambda(t_1 - t_2), \quad t_2 \geq t_1 \geq 0,$$

for some $\lambda < \lambda_1$, where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$ (see [11], [75]).

In our opinion, Theorems 1.1 and 1.2 are important for the following reasons. If we additionally assume that W is even, namely

$$W(-t) = W(t), \quad t \in \mathbb{R}, \quad (1.11)$$

by means of these theorems, we can derive the existence of various sign-changing entire solutions for the problem

$$\Delta u = W'(u), \quad x \in \mathbb{R}^n. \quad (1.12)$$

This can be done by first establishing existence of a positive solution in a suitable large “fundamental” domain $\Omega_F \subset \mathbb{R}^n$, with Dirichlet boundary conditions on $\partial\Omega_F$, and then performing consecutive odd reflections to cover the entire space. In the case where

$$W(t) = \frac{1}{4}(t^2 - 1)^2, \quad t \in \mathbb{R}, \quad (1.13)$$

then (1.12) becomes the well known Allen-Cahn equation (see for instance [70]). Assuming that W is even, namely that (1.11) holds true, then (1.2) has always the trivial solution. In this regard, the purpose of estimate (1.9) is twofold: In the case where Ω_F is bounded, it ensures that the solution of (1.2) (on Ω_F), provided by Theorem 1.2, is nontrivial. The situation of unbounded domains Ω_F can be treated by exhausting them by an increasing (with respect to inclusions) sequence $\{\Omega_n\}$ of bounded ones, each containing the same ball $B_{R'}(x_0)$, and a standard compactness argument, making use of (1.3) together with elliptic estimates and a Cantor type diagonal argument. The fact that the region of validity of estimate (1.9) increases, as $n \rightarrow \infty$, rules out the possibility of subsequences of the (chosen) solutions u_n of (1.2)_n on Ω_n converging, uniformly in compact subsets of Ω_F , to the trivial solution of (1.2) on Ω_F . Another approach for excluding this last scenario can be found in the proof of Theorem 1.3 in [28], based on a similar relation to (2.36) below (see also [44] and [70]). In this fashion, one can construct sign-changing solutions of (1.12) that belong to the following two main categories:

- “saddle solutions” which vanish on the Simons cone if n is even (see [28], [40], and [70]). (Some care is required near the vertex of the cone, see [41], [53]). The exponential estimate (1.5), in the case where Ω_F is the Simons cone, says that the corresponding saddle solution converges to μ as $|x|$ approaches infinity along certain directions. Analogous solutions exist in odd dimensions, for example when $n = 3$ one can show that there exists a solution which vanishes on all coordinate planes.
- “lattice solutions” which include solutions that are periodic in each variable x_i with period L_i , provided that L_i , $i = 1, \dots, n$, are sufficiently large (see [9], [48]).

For other types of solutions, including solutions that have saddle structure in some coordinates and are periodic in the remaining ones, the so called “tick saddle” solutions, we refer the interested reader to the introduction of [50].

A completely different approach to the construction of sign-changing solutions of (1.12), mainly applied for potentials satisfying (a), (c), and (1.11) (the typical representative being (1.13)), is based on the implementation of an infinite dimensional Lyapunov–Schmidt reduction argument, see [43], [44], [70], and the references therein. This approach produces solutions with less (or even without any) symmetry but is technically more involved.

Our Theorem 1.2 can also be used to construct multiple positive solutions of (1.2), using estimate (1.9) to make sure that they are distinct, see Section 3 below.

Let us mention that for a class of potentials, including (1.13), the dependence of the set of solutions of (1.2), in one space dimension, on the size of the interval was studied for the first time in [29].

The outline of the paper is as follows: In Section 2, we will present the proof of our main result using two different approaches, both based on a special case of a radial lemma that we prove in Subsection 2.1. In Section 3, we will show how our main result can be used to produce multiple positive solutions of (1.2) to generalize an old result of P. Hess from 1981, where nonlinear eigenvalue problems were considered. In Section 4, we study the size of the boundary layer of global minimizers of the corresponding singular perturbation problem. In Appendix A, for completeness purposes, we will state some useful lemmas that we will use in this article.

Remark 1.4. We recently found the paper [9], where it is stated that G. Fusco, in work in progress, has been able to remove the corresponding monotonicity assumption to (b) from the vector-valued Allen-Cahn type equation that was treated in [5]. After the first version of the current paper was completed, we were informed by G. Fusco that himself, F. Leonetti and C. Pignotti are working in a paper where, using the same technique developed for the vector case, they also extend the main result in [50] to more general potentials, without assuming (b). As we understand, the assumption (c) is still required in their approach.

2. PROOF OF THE MAIN RESULT

2.1. Minimizers of the energy functional on large balls. In this subsection, we will prove two lemmas concerning the asymptotic behavior of the minimizing (of the associated energy) solutions of (1.2) over large balls as their radius tends to infinity. The first one is essential for the proof of Theorem 1.2, and refines a result of P. Clément and G. Sweers [32]. The latter result is quite useful, and has been previously applied in singular perturbation problems (see [37], [58], and [60]). The second lemma, an extension of the first, is of independent interest and in particular allows for $W'(0)$ to be positive. Even though the first lemma is a special case of the second, we felt that it would be more instructive and more convenient for the reader to present them separately, since the more general second lemma is not needed for the proof of Theorem 1.2 and can be skipped at first reading.

The following is our first lemma, which is motivated from Lemma 2 in [58] and Lemma 2.2 in [60], whose origins can be traced back to [31, 32]. In these works, the weaker relation (2.11) below was established, which implies that assertion (2.3) holds *but* with constant D possibly *diverging as* $n \rightarrow \infty$ (see also Remark 2.2 below). Our improvement turns out to have interesting consequences in the study of the boundary layer of solutions of singular perturbation problems of the form (3.9) below, with $\lambda = \varepsilon^{-1} \rightarrow \infty$, see Remark 3.3 and Section 4 below.

Lemma 2.1. Assume that $W \in C^2$ satisfies condition (a'). Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.7). There exists a positive constant $R' > D$, depending only on ϵ , D , W and n , such that there exists a global minimizer u_R of the energy functional

$$J(v; B_R) = \int_{B_R} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dx, \quad v \in W_0^{1,2}(B_R), \quad (2.1)$$

which satisfies

$$0 < u_R(x) < \mu, \quad x \in B_R, \quad (2.2)$$

and

$$\mu - \epsilon \leq u_R(x), \quad x \in \bar{B}_{(R-D)}, \quad (2.3)$$

provided that $R \geq R'$. (If necessary, we assume that W is extended linearly outside of a large compact interval so that the above functional is well defined; clearly this modification does not affect the assertions of the lemma).

Proof. Under our assumptions on W , it is standard to show the existence of a global minimizer $u_R \in W_0^{1,2}(B_R)$ satisfying

$$0 \leq u_R(x) \leq \mu \quad \text{a.e. in } B_R, \quad (2.4)$$

see [50], [72], and Lemma A.1 herein (applied to the minimizing sequence converging, weakly in $W_0^{1,2}(B_R)$, to u_R). (The upper bound in (2.4) can also be derived from Lemma A.2 below, see also the second proof of Theorem 1.2). By standard elliptic regularity theory [52], this minimizer is a smooth solution, in $C^2(\bar{B}_R)$, of

$$\Delta u = W'(u) \quad \text{in } B_R; \quad u = 0 \quad \text{on } \partial B_R. \quad (2.5)$$

By the strong maximum principle (see for example Lemma 3.4 in [52]), via (2.4) and (2.5), we deduce that $u_R(x) < \mu$, $x \in B_R$, and that either u_R is identically equal to zero or $u_R(x) > 0$, $x \in B_R$ (recall that assumption **(a')** implies that $W'(0) \leq 0$ and $W'(\mu) = 0$).

By adapting an argument from Section 4 in [70] (see also Theorem 1.13 in [72]), we will show that u_R is nontrivial, provided that R is sufficiently large (depending only on W and n). (This is certainly the case when $W'(0) < 0$). It is easy to cook up a test function, and use it as a competitor, to show that there exists a positive constant C_1 , depending only on W and n , such that

$$J(u_R; B_R) \leq C_1 R^{n-1}, \quad \text{say for } R \geq 2. \quad (2.6)$$

(Plainly construct a function which interpolates smoothly from μ to 0 in a layer of size 1 around the boundary of B_R and which is identically equal to μ elsewhere, see also (2.35) below). In fact, as in Proposition 1 in [1] (see also [59]), it can be shown that

$$J(u_R; B_K) \leq \tilde{C}_1 K^{n-1} \quad \forall K < R, \quad R \geq 2, \quad (2.7)$$

where the constant $\tilde{C}_1 > 0$ depends only on W and n (see also Remark 2.9, and the arguments leading to relation (2.36) below). On the other hand, the energy of the trivial solution is

$$J(0; B_R) = \int_{B_R} W(0) dx = C_2 R^n,$$

where $C_2 > 0$ depends only on W , n . From (2.6), and the above relation, we infer that u_R is certainly not identically equal to zero for

$$R \geq C_1 C_2^{-1} + 2.$$

We thus conclude that (2.2) holds. (In the above calculation, we relied on the fact that **(a')** implies that $W(0) > 0$; in this regard, see Remark 2.7 below).

Since $u_R \in C^2(\bar{B}_R)$ is strictly positive in the ball B_R , by (2.5) and the method of moving planes [24, 51], we infer that u_R is radially symmetric and decreasing, namely

$$u'_R(r) < 0, \quad r \in (0, R), \quad (2.8)$$

(with the obvious notation). In this regard, keep in mind that if $v \in W_0^{1,2}(B_R)$ is nonnegative, then its Schwarz symmetrization $v^* \in W_0^{1,2}(B_R)$, which is radially symmetric and decreasing, satisfies $J(v^*; B_R) \leq J(v; B_R)$ (see for example [25] and the references therein). We note

that, since u_R is a global minimizer and thus stable (in the usual sense, as described in Remark 2.12 below), the radial symmetry of u_R can also be deduced as in Lemma 1.1 in [3] (see also the related references in the proof of Lemma 2.2 below). In fact, the monotonicity property (2.8) can be alternatively derived by arguing as in Lemma 2 in [27], making use of the stability of the radial solution u_R (see also the proof of Lemma 2.2 below, and Lemma 1 in [2]). Now, relation (2.6) and the nonnegativity of W clearly imply that

$$\int_{B_R \setminus B_{\frac{R}{2}}} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(u_R) \right\} dx \leq C_1 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2. \quad (2.9)$$

Hence, by the mean value theorem and the radial symmetry of u_R , there exists a $\xi \in (\frac{R}{2}, R)$ such that

$$\left\{ \frac{1}{2} [u'_R(\xi)]^2 + W(u_R(\xi)) \right\} |B_R \setminus B_{\frac{R}{2}}| \leq C_1 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

i.e.,

$$\frac{1}{2} [u'_R(\xi)]^2 + W(u_R(\xi)) \leq C_3 R^{-1}, \quad R \geq C_1 C_2^{-1} + 2, \quad (2.10)$$

where the positive constant C_3 depends only on W and n (for simplicity in notation, we have suppressed the obvious dependence of ξ on R). Hence, from assumption (a'), and relations (2.8), (2.10), we obtain that

$$u_R \rightarrow \mu, \quad \text{uniformly in } \bar{B}_{\frac{R}{2}}, \quad \text{as } R \rightarrow \infty. \quad (2.11)$$

In the sequel, we will prove that the stronger property (2.3) holds true.

For future reference, we note here that

$$[u'_R(R)]^2 \rightarrow 2W(0) \quad \text{as } R \rightarrow \infty. \quad (2.12)$$

Indeed, let

$$E_R(r) = \frac{1}{2} [u'_R(r)]^2 - W(u_R(r)), \quad r \in (0, R). \quad (2.13)$$

Thanks to (2.5), we find that

$$E'_R(r) = u''_R u'_R - W'(u_R) u'_R = -\frac{n-1}{r} (u'_R)^2, \quad r \in (0, R). \quad (2.14)$$

So,

$$E_R(R) = E_R(\xi) - \int_{\xi}^R \frac{n-1}{r} (u'_R)^2 dr, \quad (2.15)$$

where $\xi \in (\frac{R}{2}, R)$ is as in (2.10). Now, observe that (2.9) and the nonnegativity of W imply that

$$\int_{\xi}^R r^{n-1} (u'_R)^2 dr \leq C_4 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

with C_4 depending only on W and n . In turn, the above estimate clearly implies that

$$\int_{\xi}^R (u'_R)^2 dr \leq 2^{n-1} C_4, \quad R \geq C_1 C_2^{-1} + 2,$$

and it follows that

$$\int_{\xi}^R \frac{n-1}{r} (u'_R)^2 dr \leq 2^n C_4 (n-1) R^{-1}, \quad R \geq C_1 C_2^{-1} + 2. \quad (2.16)$$

The claimed relation (2.12) follows readily from (2.10), (2.13), (2.15), and (2.16). In fact, we have shown that $R|E_R(R)|$ remains uniformly bounded as $R \rightarrow \infty$.

We also consider the following family of functions

$$U_R(s) = u_R(R - s), \quad s \in [0, R]. \quad (2.17)$$

We claim that

$$U_R \rightarrow \mathbf{U}, \text{ uniformly on compact intervals of } [0, \infty), \text{ as } R \rightarrow \infty, \quad (2.18)$$

where \mathbf{U} is as in (1.8).

In view of (2.5), we get

$$U_R'' - \frac{n-1}{R-s}U_R' - W'(U_R) = 0, \quad s \in (0, R).$$

Making use of (2.2), the above equation, elliptic estimates [52], Arzela-Ascoli's theorem, and a standard diagonal argument, passing to a subsequence $R_i \rightarrow \infty$, we find that

$$U_{R_i} \rightarrow V \text{ and } U'_{R_i} \rightarrow V', \text{ uniformly on compact intervals of } [0, \infty), \text{ as } i \rightarrow \infty, \quad (2.19)$$

where $V \in C^2[0, \infty)$ is nonnegative and satisfies

$$V'' = W'(V), \quad s > 0, \text{ and } V(0) = 0.$$

Moreover, by (2.12), (2.17), and (2.19), we see that

$$[V'(0)]^2 = 2W(0) > 0.$$

By the uniqueness of solutions of initial value problems of ordinary differential equations, see for example page 108 in [82], we deduce that

$$V \equiv \mathbf{U},$$

where \mathbf{U} is as in (1.8). We also used that \mathbf{U} , V are nonnegative (which implies that $\mathbf{U}'(0)$, $V'(0)$ are also nonnegative), and the relation

$$[\mathbf{U}'(0)]^2 = 2W(0), \quad (2.20)$$

which follows from the identity

$$[\mathbf{U}'(s)]^2 - [\mathbf{U}'(0)]^2 = 2 \int_0^s W'(\mathbf{U})\mathbf{U}' ds = 2W(\mathbf{U}(s)) - 2W(0), \quad s \geq 0,$$

and the fact that $\mathbf{U}(s) \rightarrow \mu$ as $s \rightarrow \infty$, recalling that $W(\mu) = 0$ (otherwise, $\mathbf{U}'(s)$ would tend to a nonzero number and in turn $|\mathbf{U}(s)|$ to infinity, as $s \rightarrow \infty$). Moreover, by the uniqueness of the limiting function, we infer that the limits in (2.19) hold for *all* $R \rightarrow \infty$. Consequently, the claimed relation (2.18) holds.

Having (2.12), (2.18) at our disposal, we can now proceed to the proof of (2.3). Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.7). By virtue of (1.8), (1.10), and (2.18), there exists a sufficiently large R' , depending only on ϵ , D , W , n , such that $U_R(D) \geq \mu - \epsilon$, and all the previous relations continue to hold, for $R > R'$. In other words, via (2.17), we have that

$$u_R(R - D) = U_R(D) \geq \mu - \epsilon, \quad R > R'. \quad (2.21)$$

The fact that u_R is radially decreasing, recall (2.8), and the above relation imply the validity of (2.3).

The proof of the lemma is complete. \square

Remark 2.1. Our assumptions on the behavior of W near its global minimum at μ are quite weak, and in fact even allow for the potential W to have C^∞ contact with zero at the point μ , that is $W^{(i)}(\mu) = 0$, $i \geq 1$. This degeneracy translates into the absence of decay rates for the convergence of the “inner” approximate solution $\mathbf{U}(R - |x|)$ (in the sense of singular perturbation theory, see [47] and the related references that can be found in Remark 4.5 below), where \mathbf{U} is as described in (1.8), to the “outer” one μ , away from the boundary of B_R , as $R \rightarrow \infty$ (see also the discussion leading to (3.10) below). This is the main reason why we have not attempted to apply a perturbation argument, see for instance [47] and the related references in Remark 4.5 below, in order to study the asymptotic behavior of u_R as $R \rightarrow \infty$.

If $W''(\mu) > 0$, then the convergence of \mathbf{U} to μ is of order $e^{-\sqrt{W''(\mu)}s}$ as $s \rightarrow \infty$ (by the stable manifold theorem, see [33]), and one can effectively interpolate between the outer and inner approximations in order to construct a smooth global approximation that is valid throughout B_R .

Remark 2.2. By the well known relations $|B_R| = c_n R^n$, $|\partial B_R| = n c_n R^{n-1}$, $R > 0$, $n \geq 2$, for some explicit constants c_n (independent of R), where $|\partial B_R|$ denotes the $(n-1)$ -dimensional measure of ∂B_R , we find that

$$\frac{|\partial B_R|}{|B_R \setminus B_{\frac{R}{2}}|} = \frac{n 2^n}{2^n - 1} R^{-1}, \quad R > 0.$$

We deduce that the constant R' in Lemma 2.1 diverges (at least linearly) as $n \rightarrow \infty$ (see in particular the relations leading to (2.10)).

Remark 2.3. If in addition to (a') we assume that there exists some $d \in (0, \mu)$ such that

$$W'(t) \leq 0, \quad t \in (\mu - d, \mu),$$

(note that this is very natural), then relation (2.3) can alternatively be shown, starting from (2.21), *without* assuming knowledge of (2.8), as follows: Assuming, without loss of generality, that $2\epsilon < d$, by adapting the proof of Lemma A.1 below, we can find a radial $\tilde{u} \in W^{1,2}(B_{R-D})$ such that

$$J(\tilde{u}; B_{R-D}) \leq J(u_R; B_{R-D}), \quad \tilde{u}(R-D) = u_R(R-D), \quad \text{and} \quad \tilde{u}(x) \in [\mu - \epsilon, \mu], \quad x \in \bar{B}_{R-D}.$$

Thus, the function

$$\hat{u}(x) = \begin{cases} \tilde{u}(x), & x \in B_{R-D}, \\ u_R(x), & x \in B_R \setminus B_{R-D}, \end{cases}$$

belongs in $W_0^{1,2}(B_R)$ and is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$ (since $J(\hat{u}; B_R) \leq J(u_R; B_R)$). In particular, it is smooth, radial (and by virtue of its construction), and solves (2.5). It follows from Lemma 3.1 in [57], which is in the spirit of Lemma A.2 below, that the function $u_R - \hat{u}$ is either strictly positive, strictly negative, or identically equal to zero in B_R , and obviously the latter case occurs. For completeness purposes, as well as for future reference, we will draw the same conclusion by an alternative and, to our opinion, more elementary approach: The function

$$v \equiv u_R - \hat{u}$$

solves the linear equation

$$\Delta v + Q(x)v = 0, \quad x \in B_R,$$

where

$$Q(x) = \begin{cases} \frac{W'(\hat{u}(x)) - W'(u_R(x))}{u_R(x) - \hat{u}(x)}, & \text{if } \hat{u}(x) \neq u_R(x), \\ -W''(u_R(x)), & \text{if } \hat{u}(x) = u_R(x). \end{cases}$$

On the other hand, since

$$v(x) = 0, \quad x \in B_R \setminus B_{(R-D)},$$

and $Q \in L^\infty(B_R)$, the unique continuation principle (see for instance [55]) yields that

$$v(x) = 0, \quad x \in B_R.$$

(In this simple case of radial symmetry, we can also make use of the uniqueness theorem of ordinary differential equations to show that $v \equiv 0$). Therefore, estimate (2.3) holds. We remark that, if W was *strictly* decreasing in $(\mu - d, \mu)$, then (2.3) follows at once from the general lemma in [7] (see Lemma A.3 below) and (2.21).

The approach that we just presented makes only partial use of the radial symmetry of the problem (in order to establish (2.21)), and may be applied to extend some results in [37] to the general case (without radial symmetry). Moreover, it can be applied to the study of vectorial global minimizers of energy functionals, as those appearing in Lemma A.3 below, over B_R . In this case, it is known that global minimizers are radial, see [63], but monotonicity properties do not hold in general.

Remark 2.4. If $W \in C^{2,\alpha}(\mathbb{R})$, $0 < \alpha < 1$, satisfies **(a')** and

$$W''(t) \geq 0 \quad \text{for } \mu - t > 0 \text{ small}, \quad (2.22)$$

then Theorem 2 in [79] tells us that there exists a $\delta_1 \in (0, \mu)$ such that (2.5) has a unique solution such that

$$\max_{x \in \bar{B}_R} u(x) \in (\mu - \delta_1, \mu) \text{ and } -\mu < u(x) < \mu, \quad x \in B_R,$$

where $R > 0$. In view of (2.2) and (2.3), which hold for all global minimizers (with the same R'), it follows that there exists a unique global minimizer of (2.1) if R is sufficiently large.

If the stronger assumption $W''(\mu) > 0$ holds (in other words **(c)**), then a simple proof of the uniqueness of the global minimizer, for large R , can be given as follows: One first shows that if a solution of (2.5) satisfies (2.2), (2.3), and (2.18) (recall (2.17)), then it is *asymptotically* stable for large $R > 0$. The key for this is the fact that **U** itself is asymptotically stable, which follows readily from the fact that it is increasing and **(c)**, via Sturm's theorem. Suppose then that u_1 and u_2 are two distinct global minimizers of (2.1). By the proof of Lemma 2.1, they satisfy (2.2), (2.3), and (2.18) uniformly (independent of the choice of minimizers) as $R \rightarrow \infty$. Thanks to Lemma 3.1 in [57] (see also Lemma A.2 herein), without loss of generality, we may assume that $u_1(x) < u_2(x)$, $x \in B_R$. On the other hand, by the mountain pass theorem or the theory of monotone dynamical systems (see [42], [64] respectively, and Section 3 herein), we infer that there exists an *unstable* solution \hat{u}_1 of (2.5) such that $u_1(x) < \hat{u}_1(x) < u_2(x)$, $x \in B_R$. In particular, the unstable solution enjoys the asymptotic behavior of global minimizers, as $R \rightarrow \infty$, and thus is asymptotically stable (by our previous discussion); a contradiction. A related uniqueness proof, based on a dynamical systems argument (but not of monotone nature), can be found in [2], see also [12].

Remark 2.5. If in addition to **(a')** we assume that $W'(0) = 0$ and $W''(0) < 0$, then (2.5) admits a nontrivial positive solution, which is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$, as long as $R > R_c$, where

$$R_c = \sqrt{-\frac{\lambda_1}{W''(0)}},$$

and λ_1 denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(B_1)$ (an analogous result holds for (3.9) below). If we further assume that

$$W'(t) \geq W''(0)t, \quad t \geq 0,$$

then (2.5), for $R \in (0, R_c)$, has no positive solution. These assertions can be proven by adapting the proof of Lemma 2.1 in [44]. If in addition to **(a')**, $W'(0) = 0$ and $W''(0) < 0$, we impose some extra assumptions on W , which are satisfied for example by $W(u) = u(u-a)(u-\mu)$ with $0 < a < \mu/2$ not too small, it was shown in [83] that there exists exactly one solution for $R = R_c$ and exactly two for $R > R_c$ (see also [76] and the references therein).

Remark 2.6. By (2.9), via the coarea formula (see [46]), it follows that there exists a $\xi_R \in (\frac{R}{2}, R)$ such that

$$\int_{\partial B_{\xi_R}} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(u_R) \right\} dx \leq 2C_1 R^{n-2}, \quad R \geq C_1 C_2^{-1} + 2.$$

This observation makes no use of the radial symmetry of u_R , and is motivated from [7]. In regard to the latter comment, see the clearing-out Lemma 1 in [22].

Remark 2.7. In case a C^2 potential W satisfies $W(0) = 0$ and the domain Ω has C^1 boundary, is bounded, and star-shaped with respect to some point in Ω , the well known Pohozaev identity easily implies that there does not exist a nontrivial solution of (1.2) such that $W(u(x)) \geq 0$, $x \in \Omega$ (see for instance relation (11) in [10], a reference which is in accordance with our notation). In this regard, we also refer to Chapter 4 in [77]. Actually, relation (11) in the latter reference holds true for the elliptic system that corresponds to (1.2), and an analogous nonexistence result can be deduced in that situation as well.

Remark 2.8. Under the stronger assumptions **(a)** (or more generally **(a')**), **(b)**, and **(c)**, considered in [50] (recall the introduction herein), motivated from the proof of Lemma 3 in [61], we can give a streamlined proof of relation (2.11) as follows: Firstly, note that, thanks to **(a')** and **(c)**, there exists a positive constant c_0 such that

$$W(t) \geq c_0(\mu - t)^2, \quad 0 \leq t \leq \mu.$$

Then, bounds (2.2), (2.6), and the above relation yield that

$$\int_{B_R} (\mu - u_R)^2 dx \leq c_1 R^{n-1}, \tag{2.23}$$

where the positive constant c_1 depends only on W and n . Now, note that assumption **(b)**, bound (2.2), and the equation in (2.5), imply that the function $\mu - u_R$ is subharmonic in B_R , and thus we have

$$\Delta(\mu - u_R)^2 \geq 0 \quad \text{in } B_R,$$

in other words $(\mu - u_R)^2$ is also subharmonic. Consequently, by (2.23) and the mean value property of subharmonic functions (see Theorem 2.1 in [52]) together with a simple covering argument, see also the general Theorem 9.20 in [52], we deduce that

$$\max_{\bar{B}_{\frac{R}{2}}}(\mu - u_R)^2 \leq c_2 R^{-n} \int_{B_R} (\mu - u_R)^2 dx \leq c_3 R^{-1},$$

where the positive constants c_2, c_3 depend only on W and n . The latter inequality clearly implies the validity of (2.11).

The above argument makes no use of the fact that u_R is radially symmetric. Moreover, it works equally well if $W(t) \geq c(\mu - t)^p$, $t \in [0, \mu]$, for some constants $c > 0$ and $p > 2$. Hopefully, this approach may be adapted to simplify the arguments of Section 6 in [6], where the De Giorgi oscillation lemma for subharmonic functions was employed instead of the mean value property.

One might be tempted to apply the Harnack inequality (see Theorem 8.20 in [52]) to the equation

$$\Delta(\mu - u_R) + \frac{W'(u_R)}{(\mu - u_R)}(\mu - u_R) = 0, \quad x \in B_R,$$

but, unfortunately, the constant involved depends on R .

An extension of Lemma 2.1 can be shown, allowing the possibility $W'(0) \geq 0$, provided that the potential W satisfies:

(a''): There exist constants $\mu_- \leq 0$ and $\mu > 0$ such that

$$0 = W(\mu) < W(t), \quad t \in [\mu_-, \mu], \quad W(t) \geq 0, \quad t \geq \mu,$$

$$W(2\mu_- - t) \geq W(t), \quad t \in [\mu_-, \mu].$$

Note that (a'') reduces to (a') when $\mu_- = 0$. Below, we state such a result which seems to be new and of independent interest.

Lemma 2.2. Assume that $W \in C^2$ satisfies condition (a''). Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.7). Then, there exists a positive constant $R' > D$, depending only on ϵ, D, W , and n , such that there exists a global minimizer u_R of the energy functional in (2.1) which satisfies (2.2) and (2.3), provided that $R \geq R'$. (As before, we assume that W has been appropriately extended outside of a large compact interval). (We have chosen to keep some of the notation from Lemma 2.1).

Proof. The existence of a minimizer u_R , which solves (2.5), and satisfies

$$\mu_- < u_R(x) < \mu, \quad x \in B_R,$$

follows as in the proof of Lemma 2.1. The main difference with the proof of Lemma 2.1 is that the above relation does not exclude the possibility of the minimizer u_R taking non-positive values. In particular, the method of moving planes (see [24], [51]) is not applicable in order to show that u_R is radially symmetric and decreasing. (Nevertheless, the radial symmetry of nonnegative solutions of (2.5) can still be shown by the method of moving planes, see [71] and the references therein). Not all is lost however, thanks to Lemma 1.1 in [3] (see also Corollary II.10 in [63], and [37], [66]). Indeed, as we have already remarked in the proof of Lemma 2.1, the stability of u_R (as a global minimizer) implies that it is radially symmetric. Next, we will show that $u_R(r)$ is a decreasing function of r , namely that (2.8) holds true. In view of (2.10), which still holds for the case at hand (by virtue

of radial symmetry alone), it suffices to show that $u'_R(r) \neq 0$, $r \in (0, R]$. We will follow the part of the proof of Lemma 2 in [27] which dealt with problem (1.12) with $n \geq 3$, and in fact show that it continues to apply for $n \leq 2$. To this end, we have not been able to adapt the approach of Lemma 1 in [2], which basically consists in multiplying (2.26) below by $V^+ \equiv \max\{V, 0\} \in W^{1,2}(B_R)$ and integrating the resulting identity by parts over B_R , since in the problem at hand $V(R) = u'_R(R)$ may be positive. Let

$$V \equiv u'_R,$$

and suppose, to the contrary, that $V(R_0) = 0$ for some $R_0 \in (0, R]$. We will show that the function

$$\tilde{V}(r) = \begin{cases} V(r), & r \in [0, R_0], \\ 0 & r \in [R_0, R], \end{cases} \quad (2.24)$$

belonging in $W_0^{1,2}(B_R)$, satisfies

$$\int_{B_R} \left\{ |\nabla \tilde{V}|^2 + W''(u_R) \tilde{V}^2 \right\} dx < 0, \quad (2.25)$$

which clearly contradicts the stability of u_R . Differentiating (2.5) with respect to r , we arrive at

$$-\Delta V + W''(u_R)V + \frac{n-1}{r^2}V = 0, \quad x \in B_R \setminus \{0\}. \quad (2.26)$$

Let ζ be a smooth function such that

$$\zeta(t) = \begin{cases} 0, & t \in [0, 1], \\ 1, & t \in [2, \infty). \end{cases}$$

Multiplying (2.26) by $\zeta\left(\frac{r}{\varepsilon}\right)V(r)$, with $\varepsilon > 0$ small, and integrating the resulting identity by parts over B_{R_0} (recall that $V(R_0) = 0$), we find that

$$\int_{B_{R_0}} \left\{ \zeta\left(\frac{r}{\varepsilon}\right) |\nabla V|^2 + \frac{1}{\varepsilon} V \zeta'\left(\frac{r}{\varepsilon}\right) \left(\frac{x}{r} \cdot \nabla V\right) + \zeta\left(\frac{r}{\varepsilon}\right) W''(u_R) V^2 + \zeta\left(\frac{r}{\varepsilon}\right) \frac{n-1}{r^2} V^2 \right\} dx = 0. \quad (2.27)$$

Note that

$$\left| \int_{B_{R_0}} \frac{1}{\varepsilon} V \zeta'\left(\frac{r}{\varepsilon}\right) \left(\frac{x}{r} \cdot \nabla V\right) dx \right| \leq C \varepsilon^{-1} \int_{\varepsilon}^{2\varepsilon} r^{n-1} dr \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

since the constant $C > 0$ does not depend on ε . (Note that we have silently assumed that $N \geq 2$, since in the case $N = 1$ we can plainly multiply (2.26) by V and then integrate by parts over $(-R_0, R_0)$). So, letting $\varepsilon \rightarrow 0$ in (2.27), and employing Lebesgue's dominated convergence theorem (see for instance page 20 in [46]), it readily follows that

$$\int_{B_{R_0}} \left\{ |\nabla V|^2 + W''(u_R) V^2 + \frac{n-1}{r^2} V^2 \right\} dx = 0,$$

where in order to obtain the last term we used that $|V(r)| \leq C'r$, $r \in [0, R]$, with constant $C' > 0$ depending only on R (keep in mind that $u_R \in C^2[0, R]$ with $u_R''(0) = \frac{1}{n}W'(u_R(0))$, see for instance page 72 in [82]). From the above relation, via (2.24), we get (2.25). We have thus arrived at the desired contradiction. Consequently, the monotonicity relation (2.8) also

holds for the more general case at hand. The rest of the argument follows word by word the proof of Lemma 2.1, and is therefore omitted.

The proof of the lemma is complete. \square

Remark 2.9. Suppose that u_R is as in Lemma 2.1 or Lemma 2.2, and E_R as defined in (2.13). From (2.14), it follows that

$$E_R(r) < E_R(0) = -W(u_R(0)) < 0, \quad r \in [0, R],$$

i.e.,

$$\frac{1}{2}[u'_R(r)]^2 < W(u_R), \quad r \in (0, R], \quad (2.28)$$

recall (a') and that $u'_R(0) = 0$, see also Remark 4 in [2] for a related discussion. In passing, we note that every bounded solution of (1.12) satisfies

$$\frac{1}{2}|\nabla u|^2 \leq W(u), \quad x \in \mathbb{R}^n,$$

provided that W is nonnegative. The proof of this gradient bound, originally due to L. Modica, is much more complicated than that of its radially symmetric counterpart (2.28). We refer the interested reader to Lemma 4.1 in [30], and to the older references that can be found in [4]. In turn, making use of the gradient bound (2.28), we can establish the monotonicity formula

$$\frac{d}{dr} \left(\frac{1}{r^{n-1}} \int_{B_r} \left\{ \frac{1}{2}|\nabla u_R|^2 + W(u_R) \right\} dx \right) > 0, \quad r \in (0, R), \quad (2.29)$$

see [4] for a modern approach as well as the older references therein. (A similar monotonicity formula holds true for solutions of (1.12), and a weaker one (with the exponent $n-1$ replaced by $n-2$) holds in the case of the corresponding systems, see [4]). Now, making use of (2.6) and the above relation, we find that

$$\frac{1}{K^{n-1}} \int_{B_K} \left\{ \frac{1}{2}|\nabla u_R|^2 + W(u_R) \right\} dx < C_1 \quad \forall K \in (0, R), \quad R \geq 2.$$

We have therefore provided a proof (of a sharper version) of (2.7). It also follows from (2.29) that $R^{1-n}J(u_R; B_R)$ remains bounded from below by some positive constant, as $R \rightarrow \infty$ (compare with (2.6)). If $W''(\mu) > 0$, making use of (2.18), it is not hard to determine the constant to which $R^{1-n}J(u_R; B_R)$ converges as $R \rightarrow \infty$, recall the last part of Remark 2.1.

Remark 2.10. Here, for completeness, we sketch an argument related to the proof of Lemma 2.2. By (2.2), elliptic estimates (see [52]), and a standard compactness argument, it follows readily that u_R converges, up to a subsequence $R_i \rightarrow \infty$, uniformly on compact subsets of \mathbb{R}^n to a radially symmetric solution U of (1.12) such that $0 \leq U(x) \leq \mu$, $x \in \mathbb{R}^n$. Moreover, this solution is a global minimizer of (1.12) in the sense of (2.37) below, with $\Omega = \mathbb{R}^n$, see [28], [57]. On the other hand, it is known that (1.12), for any $W \in C^2$, does not have nonconstant bounded, radial, global minimizers (see [81]). Obviously $U \equiv \mu$ (recall (a')) and, by the uniqueness of the limit, the convergence is for all $R \rightarrow \infty$. We conclude that, given any $K > 1$, we have $u_R \rightarrow \mu$, uniformly in B_K , as $R \rightarrow \infty$. The advantage of this approach is that it continues to work when (2.5) is replaced by $\Delta u = F_R(|x|, u)$, with a suitable $F_R(|x|, u)$ which converges uniformly over compact sets of $[0, \infty) \times \mathbb{R}$ to a C^1 function $F(u)$ (the point being that $\frac{d}{dr}F_R(r, u)$ may be negative somewhere, and (2.8) may fail in B_R).

The following corollary is a simple consequence of the maximum principle.

Corrolarry 2.1. If $W''(\mu) > 0$, then the solutions provided by Lemmas 2.1 and 2.2 satisfy

$$\mu - u_R(r) \leq C_5 e^{-C_6(R-r)}, \quad r \in [0, R - 2D] \quad \text{for } R \geq R',$$

and some positive constants C_5, C_6 , depending on W and n .

Proof. Let $\varphi \equiv \mu - u_R$, where u_R is as in Lemma 2.1 or 2.2. By virtue of (a'), (2.2), and (2.3), we can choose ϵ sufficiently small such that that

$$W'(u_R(x)) \leq \frac{W''(\mu)}{2} (u_R(x) - \mu), \quad x \in B_{(R-D)},$$

provided that $R \geq R'$, where D, R' are as in the previously mentioned lemmas (having increased the value of R' , if necessary, but still depending only on ϵ, D, W , and n). It follows from (2.5) that

$$-\Delta\varphi + \frac{W''(\mu)}{2}\varphi \leq 0 \quad \text{in } B_{(R-D)}, \quad R \geq R'.$$

Now, the desired assertion of the corollary follows from a standard comparison argument, see Lemma 2 in [21] or Lemma 4.2 in [47].

The proof of the corollary is complete. \square

Remark 2.11. A special case of Theorem 2.1 in [35] shows that the assertion of Corollary 2.1 above can be considerably refined to

$$\lim_{R \rightarrow \infty} R^{-1} \ln(\mu - u_R(Rs)) = -(1-s)\sqrt{W''(\mu)}, \quad \forall s \in [0, 1],$$

see also [14].

2.2. Proof of Theorem 1.2. Once Lemma 2.1 is established, the proof of Theorem 1.2 proceeds in a rather standard way. We will present two different approaches, and leave it to the reader's personal taste. The first approach is based on the method of upper and lower solutions, while the second is based on variational arguments.

First proof of Theorem 1.2: We will adapt an argument from the proof of Theorem 2.1 in [39], and prove existence of the desired solution to (1.2) by the method of upper and lower solutions (see for instance [64], [74]). Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.7), and R' be the positive constant, depending only on ϵ, D, W , and n , that is described in Lemma 2.1. Suppose that Ω contains a closed ball of radius R' . We use $\bar{u}(x) \equiv \mu$, $x \in \Omega$, as an upper solution (recall that $W'(\mu) = 0$), and as lower solution the function

$$\underline{u}_P(x) \equiv \begin{cases} u_{\text{dist}(P, \partial\Omega)}(x - P), & x \in B_{\text{dist}(P, \partial\Omega)}(P), \\ 0, & x \in \Omega \setminus B_{\text{dist}(P, \partial\Omega)}(P), \end{cases} \quad (2.30)$$

for some $P \in \Omega_{R'}$ (considered fixed for now), where u_R is as in Lemma 2.1 (here we used that $W'(0) \leq 0$ and Proposition 1 in [15] to make sure that \underline{u}_P is a lower solution). In view of (2.2) and (2.3), keeping in mind that

$$\text{dist}(P, \partial\Omega) > R', \quad (2.31)$$

it follows that

$$\underline{u}_P(x) < \bar{u}(x) \equiv \mu, \quad x \in \Omega, \quad \text{and} \quad \mu - \epsilon < \underline{u}_P(x), \quad x \in B_{(\text{dist}(P, \partial\Omega) - D)}(P). \quad (2.32)$$

In the case where Ω is bounded, it follows immediately from the method of monotone iterations, see Theorem 2.3.1 in [74] (see also the book [53] for the corresponding elliptic

estimates that are required in case the domain has corners, and [41]), that there exists a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of (1.2) such that

$$\underline{u}_P(x) < u(x) < \bar{u}(x) \equiv \mu, \quad x \in \Omega,$$

(have in mind that the solution u depends on the choice of the center P). The same property can also be shown in the case where Ω is unbounded, by exhausting it with a sequence of bounded domains, see Theorem 2.10 in [65] (also recall our discussion following the statement of Theorem 1.2), see also [67, 69]. We have thus established the existence of a solution u of (1.2) that satisfies (1.3), and the lower bound (1.9) in the region $B_{(\text{dist}(P, \partial\Omega) - D)}(P)$ (recall (2.2), (2.3), and (2.31)), or equivalently in $P + B_{(\text{dist}(P, \partial\Omega) - D)} \supseteq P + B_{(R' - D)}$. It remains to show that the latter lower bound is valid in $\Omega_{R'} + B_{(R' - D)}$. Observe that, as we vary the point P in $\Omega_{R'}$, assuming for the moment that $\Omega_{R'}$ has a single arc-wise connected component, the functions \underline{u}_P 's continue to be lower solutions of (1.2). Consequently, by Serrin's sweeping principle (see [32, 34, 74]), we deduce that

$$\underline{u}_Q(x) < u(x), \quad x \in \Omega, \quad \forall Q \in \Omega_{R'}, \quad (2.33)$$

(see also the proof of Lemma 3.1 in [32]). The validity of the lower bound (1.9), over the whole specified domain, now follows from (2.3), (2.30), (2.31), and (2.33). In the case where the domain $\Omega_{R'}$ has numerable many arc-wise connected components, we can use the function $\max\{u_{P_i}, i = 1, \dots\}$ as a lower solution, where the u_{P_i} 's are as in (2.30) with each center P_i belonging to a different component of $\Omega_{R'}$. (We use again Proposition 1 in [15], keep in mind that the maximum is essentially chosen among finitely many functions). The case where there are denumerable many arc-wise connected components of $\Omega_{R'}$ can be treated similarly.

If $W''(\mu) > 0$, the validity of (1.5) for $x \in \Omega_{R'}$ follows at once from Corollary 2.1 and relations (2.30), (2.33). If $\text{dist}(x, \partial\Omega) \leq R'$, then plainly observe that

$$\mu - u(x) \leq \mu = \mu e^{R'} e^{-R'} \leq \mu e^{R'} e^{-\text{dist}(x, \partial\Omega)}. \quad (2.34)$$

The first proof of the theorem is complete. \square

Remark 2.12. Since it is constructed by the method of upper and lower solutions, we know that the obtained solution u is stable (with respect to the corresponding parabolic dynamics), see [64, 74], namely the principal eigenvalue of

$$-\Delta\varphi + W''(u)\varphi = \lambda\varphi, \quad x \in \Omega; \quad \varphi = 0, \quad x \in \partial\Omega,$$

is nonnegative. In the case of unbounded domains, some extra care is needed in the definition of stability, see [27, 38].

Second proof of Theorem 1.2: Assume first that Ω is bounded. As in the proof of Lemma 2.1, there exists a global minimizer u_{\min} of the energy

$$J(v; \Omega) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dx, \quad v \in W_0^{1,2}(\Omega),$$

which furnishes a classical solution of (1.2) such that $0 \leq u_{\min}(x) < \mu$, $x \in \Omega$. Again, by the strong maximum principle, either u_{\min} is identically equal to zero or it is strictly positive in Ω . We intend to show that there exists an $R_* > 0$, depending only on W and n , such that u_{\min} is nontrivial, provided that Ω contains some closed ball of radius R_* .

For the sake of our argument, suppose that u_{min} is the trivial solution. Then, motivated from Proposition 1 in [1] (see also [28], [59] and [68]), assuming without loss of generality that $\bar{B}_{R+2} \subset \Omega$ for some $R > 0$, we consider the function

$$Z(x) = \begin{cases} 0, & x \in \Omega \setminus B_{R+1}, \\ \mu(R+1-|x|), & x \in B_{R+1} \setminus B_R, \\ \mu, & x \in B_R. \end{cases} \quad (2.35)$$

Since $Z \in W_0^{1,2}(\Omega)$, from the relation $J(0; \Omega) \leq J(Z; \Omega)$, and recalling that $W(\mu) = 0$, we obtain that

$$J(0; B_{R+1}) \leq \int_{B_{R+1} \setminus B_R} \left\{ \frac{1}{2} |\nabla Z|^2 + W(Z) \right\} dx \leq C_0 R^{n-1}, \quad (2.36)$$

with C_0 depending only on W and n . In turn, the above relation implies that

$$|B_{R+1}|W(0) \leq C_0 R^{n-1},$$

which cannot hold if $R \geq R_*$ is sufficiently large, depending on W and n . Consequently, the minimizer u_{min} is nontrivial, provided that Ω contains some closed ball of radius R_* . From our previous discussion, we therefore conclude that u_{min} satisfies (1.3).

Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.7). Suppose that Ω contains a closed ball of radius R' , where R' is as in the assertion of Lemma 2.1; without loss of generality, we may assume that $R' > R_*$. Relation (1.9) now follows by applying Lemma A.2 below, over every closed ball of radius R' contained in Ω , and recalling Lemma 2.1. Note that, as in Remark 2.3, the unique continuation principle implies that

$$u_{min} \text{ minimizes } J(v; \mathcal{D}) \text{ in } v - u_{min} \in W_0^{1,2}(\mathcal{D}) \text{ for every smooth bounded domain } \mathcal{D} \subset \Omega. \quad (2.37)$$

The case where Ω is unbounded can be treated by exhausting it by an infinite sequence of bounded ones, where the above considerations apply (see also [68]). The minimizers over the bounded domains (extended by zero outside) converge locally uniformly to a solution u of (1.2) that satisfies (1.3) (the latter solution is nontrivial by virtue of the lower bound $u(x) \geq \mu - \epsilon$, $x \in B_{(R'-D)}(x_0)$ for some $x_0 \in \Omega_{R'}$, which is valid since we may assume that each one of the bounded domains contains the same closed ball $\bar{B}_{R'}(x_0)$). This solution of (1.2), on the unbounded domain Ω , found in this way, may have infinite energy but is still a global minimizer in the sense of Definition 1.2 in [57], namely satisfies (2.37). As before, it satisfies (1.9).

If $W''(\mu) > 0$, the validity of (1.5) follows at once from Corollary 2.1, Lemma A.2 below (applied on every ball $B_{\text{dist}(x, \partial\Omega)}(x)$, $x \in \Omega_{R'}$), and the observation in (2.34).

The second proof of the theorem is complete. \square

Remark 2.13. If W is as in Remark 2.4, and Ω is bounded with smooth boundary (at least C^3), in view of the latter remark and Theorem 2 in [79], the solutions of Theorem 1.2 that we found by the two different approaches are actually the same, if ϵ is chosen sufficiently small.

Remark 2.14. Assume that the domain Ω is symmetric with respect to the hyperplane $x_i = 0$. Then, since the solution of (1.2), provided by the second proof of Theorem 1.2, is a global minimizer of the associated energy (in the sense described above, in case Ω is

unbounded), it follows from Theorem II.5 in [63] (applied on symmetric bounded domains, with respect to the hyperplane $x_i = 0$, exhausting Ω) that the latter solution is symmetric with respect to this hyperplane. Note that, if in addition the domain Ω is bounded and convex in the x_i direction, this assertion holds true for *any* positive solution of (1.2) by virtue Theorem 2 in [24] (proven by the method of moving planes). Clearly, if uniqueness holds for positive solutions of (1.2) (recall Remark 3.2), these assertions follow at once (see also Remark 1.3 in [50]).

3. EXTENSIONS

Suppose that $W : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 and there are positive numbers

$$\mu_1 < \cdots < \mu_m, \quad m \geq 2,$$

such that

$$W(\mu_1) > \cdots > W(\mu_m), \quad W'(0) \leq 0, \quad W'(\mu_i) = 0, \quad i = 1, \dots, m,$$

and

$$W(t) > W(\mu_i), \quad t \in [0, \mu_i), \quad i = 1, \dots, m.$$

Note that at the points μ_i , the potential W has either minima or saddles.

Obviously, we can extend W outside of $[0, \mu_i]$, to a C^2 potential \tilde{W}_i , in such a way that condition (a') is satisfied with $\tilde{W}_i(t) - W(\mu_i)$ in place of W and μ_i in place of μ , $i = 1, \dots, m$. Next, consider any

$$\epsilon \in \left(0, \min_{i=1, \dots, m} (\mu_i - \mu_{i-1})\right), \quad (3.1)$$

with the convention that $\mu_0 = 0$, and any

$$D_i > D'_i \text{ where } D'_i \text{ solve } \mathbf{U}_i(D'_i) = \mu_i - \epsilon, \quad i = 1, \dots, m, \quad (3.2)$$

where

$$\mathbf{U}_i''(s) = W'(\mathbf{U}_i(s)), \quad s > 0; \quad \mathbf{U}_i(0) = 0, \quad \lim_{s \rightarrow \infty} \mathbf{U}_i(s) = \mu_i. \quad (3.3)$$

By means of Theorem 1.2, there exist positive numbers $R'_i > D_i$, depending only on ϵ , D_i , \tilde{W}_i , $i = 1, \dots, m$, and n , such that if Ω contains a closed ball of radius R'_i then there exists a solution u_i of

$$\Delta u = \tilde{W}'_i(u), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega, \quad (3.4)$$

satisfying

$$0 < u_i(x) < \mu_i, \quad x \in \Omega, \quad (3.5)$$

and

$$\mu_{i-1} < \mu_i - \epsilon < u_i(x), \quad x \in \Omega_{R'_i} + B_{(R'_i - D_i)}, \quad i = 1, \dots, m. \quad (3.6)$$

In view of (3.5), we conclude that u_i solves the original problem (1.2). Thus, given ϵ and D_i as in (3.1) and (3.2) respectively, if Ω contains a closed ball of radius R'' , where $R'' = \max_{i=1, \dots, m} R'_i$, we find that (1.2) has at least m positive solutions which satisfy (3.5)–(3.6). Moreover, keeping in mind Remark 2.12, we know that these solutions are stable.

These solutions may be chosen to be ordered, in the usual sense. In other words, given ϵ and D_i as in (3.1) and (3.2) respectively, there are at least m positive, stable solutions of (1.2) such that

$$u_1(x) < \cdots < u_m(x), \quad x \in \Omega, \quad 1 \leq i \leq m, \quad (3.7)$$

and (3.5)–(3.6) hold (we have chosen to keep the same notation). Indeed, the solution u_i can be captured by using the constant function μ_i as an upper solution; and the function $\max\{u_{i-1}(x), \underline{u}_P^i\}$ as lower solution, where \underline{u}_P^i is the lower solution in (2.30) but with $\tilde{W}_i(t) - W(\mu_i)$ in place of $W(t)$, $i = 1, \dots, m$, and $u_0 \equiv 0$. (We use again Proposition 1 in [15] to make sure that it is a lower solution). As in the first proof of Theorem 1.2, we can sweep with the family of lower solutions \underline{u}_Q^i , $Q \in \Omega_{R'_i}$ to extend the lower bound on u_i (due to (2.3)) from $B_{(R'_i - D_i)}(P)$ to $\Omega_{R'_i} + B_{(R'_i - D_i)}$. Moreover, the strong inequalities in (3.7) follow from the strong maximum principle. Naturally, the obtained solutions are stable (recall Remark 2.12).

We have just proven the following:

Theorem 3.1. Suppose that Ω and W are as described in this section. Let ϵ and D_i be as in (3.1) and (3.2) respectively. There exist positive constants $R'_i > D_i$, $i = 1, \dots, m$, depending only on ϵ , D_i , W and n , such that if Ω contains a closed ball of radius $R'' = \max_{i=1, \dots, m} R'_i$, then problem (1.2) has at least m stable solutions u_i , ordered as in (3.7), such that (3.5)–(3.6) hold true.

On the other hand, assuming that Ω is bounded and smooth (a C^3 boundary suffices), the theory of monotone dynamical systems (see Theorem 4.4 in [64]) guarantees the existence of at least $m - 1$ unstable solutions \hat{u}_i , $i = 1, \dots, m - 1$, of (1.2) such that

$$u_i(x) < \hat{u}_i(x) < u_{i+1}(x), \quad x \in \Omega, \quad i = 1, \dots, m - 1. \quad (3.8)$$

This can also be shown by the well known mountain pass theorem, see [42].

Remark 3.1. The extra assumptions on Ω were only required in order to apply Theorem 4.4 in [64].

In summary, we have

Theorem 3.2. Suppose that, in addition to the hypotheses of Theorem 3.1, the domain Ω is assumed to be smooth and bounded. Then, besides of the m stable solutions u_i that are provided by Theorem 3.1, there exist at least $m - 1$ unstable solutions \hat{u}_i of (1.2), ordered as in (3.8) (keep in mind (3.7)).

The above theorem extends an old result of P. Hess [54], in the context of nonlinear eigenvalue problems (which are included in our setting, see below), where the additional assumption that $W'(0) < 0$ was imposed (see also [26] for an earlier result in the case $n = 1$). In the same context, the case $W'(0) = 0$ was allowed in [42], at the expense of assuming that $W'(\mu_i) \neq 0$, $i = 1, \dots, m$, and some geometric restrictions on the domain. All these references considered nonlinear eigenvalue problems of the form

$$\Delta u = \lambda^2 W'(u), \quad x \in \mathcal{D}, \quad u(x) = 0, \quad x \in \partial \mathcal{D}, \quad (3.9)$$

where \mathcal{D} is a smooth bounded domain of \mathbb{R}^n . By stretching variables $x \mapsto \lambda^{-1}x$, assuming that $0 \in \mathcal{D}$ (this we can do without loss of generality), keeping the same notation, we are led to the equivalent problem:

$$\Delta u = W'(u), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega, \quad (3.10)$$

where $\Omega \equiv \lambda \mathcal{D}$, for $\lambda > 0$, which is plainly problem (1.2). If λ is sufficiently large, then certainly the domain Ω contains the ball $B_{R''}$, appearing in the assertion of Theorem 3.1, but not the other way around. In contrast to our approach of using upper and lower solutions, De Figueiredo in [42] obtained the corresponding stable solutions as minimizers of the

associated energy functionals (with W suitably modified outside of $[0, \mu_i]$, $i = 1, \dots, m$), and a geometric condition had to be imposed on the domain in order to ensure that they are distinct for large λ . In our case, the fact that they are distinct follows at once from (3.5) and (3.6). As we have already pointed out, in [42], the unstable solutions were constructed as mountain passes (saddle points of the energy).

Remark 3.2. It has been proven in [34] that if $W'(t) < 0$, $t \in (0, \mu)$, $W'(0) < 0$, or $W'(0) = 0$ but $W''(0) < 0$, $W'(\mu) = 0$, and $W'' \geq 0$ near μ , then (3.9) has a unique solution with values $(0, \mu)$ when λ is large, see also [12].

Remark 3.3. If \mathcal{D} is a bounded domain with boundary satisfying the interior ball condition (see [52], a sufficient condition for this is that the boundary is C^2), it follows from the proof of Theorem 1.2 that the corresponding stable solutions of (3.9), provided by Theorem 3.1, develop a boundary layer of size $\mathcal{O}(\lambda^{-1})$, as $\lambda \rightarrow \infty$, along the boundary of \mathcal{D} (see Proposition 4.1 below for more details, and compare with the proof of Theorem 1.1 in [60], as well as with Theorem 4 in [42] and Lemma 2 in [58]). Loosely speaking, the stable solutions u_i converge uniformly to μ_i on the domain \mathcal{D} excluding the strip that is described by $\text{dist}(x, \partial\mathcal{D}) \leq |\ln \lambda|^\alpha \lambda^{-1}$, $\alpha > 0$, as $\lambda \rightarrow \infty$. It follows from (3.8) that the corresponding unstable solutions of (3.9), provided by Theorem 3.2, also develop a boundary layer.

In fact, if $W''(\mu_i) > 0$, the fine structure of the boundary layer of u_i is determined by the unique solution of the problem (3.3), see [12] and Remark 4.4 below. On the other hand, the unstable solutions \hat{u}_i typically have an upward sharp spike layer on top of u_i , near the most centered part of the domain \mathcal{D} , whose fine structure is determined by the problem

$$\Delta V = W'(V + \mu_i) \text{ in } \mathbb{R}^n; \quad V(x) \rightarrow 0, \quad |x| \rightarrow \infty,$$

see [35], [36], and [56].

4. ON THE BOUNDARY LAYER OF GLOBAL MINIMIZERS OF SINGULARLY PERTURBED ELLIPTIC EQUATIONS

In this section, assuming (a'), we will prove a general result on the size of the boundary layer of solutions of (3.9), which minimize the associated energy functional, as $\lambda \rightarrow \infty$ (recall also Remark 3.3). Setting $\varepsilon = \lambda^{-1} \rightarrow 0$, gives rise to a singular perturbation problem (recall Remark 2.1).

We emphasize that, in contrast to previous results in this direction, as Theorem 1.1 in [60], here the size of the boundary layer is shown to be *independent of the dimension n* . This is due to our previous improvement over Lemma 2.2 in [60] that was made in Lemma 2.1 herein (recall the discussion preceding it, and also see Remark 4.3 below). The point is that we have not assumed any nondegeneracy on W at μ ; in the case where $W''(\mu) > 0$ or $n = 2$, the structure of the boundary layer is well understood (recall Remark 3.3). For a different possible approach to this, see Remark 4.4 below.

The main result in this section is

Proposition 4.1. Suppose that \mathcal{D} is a bounded domain in \mathbb{R}^n , $n \geq 1$, whose boundary satisfies the interior ball condition (recall Remark 3.3), and let W satisfy assumption (a'). Consider any $\epsilon \in (0, \mu)$ and $D > D'$, where D' as in (1.7). There exists a positive constant λ_* , depending only on ϵ , D , \mathcal{D} , and W , such that there exists a solution u_λ of (1.2), which minimizes the associated energy functional, satisfies (1.3) and

$$u_\lambda(x) \geq \mu - \epsilon, \quad x \in \bar{\mathcal{D}}_{(D\lambda^{-1})}, \quad (4.1)$$

provided that $\lambda \geq \lambda_*$ (recall the definition (1.6), and note that $\mathcal{D}_{(D\lambda^{-1})}$ is a connected domain for large λ). (See also the comments at the end of the assertion of Lemma 2.1)

Proof. As in the second proof of Theorem 1.2, recalling the discussion leading to (3.10), there exists a smooth solution of (3.9), which minimizes the associated energy and satisfies (1.3), provided that λ is sufficiently large, say $\lambda \geq \lambda_0$, depending not just on W but this time also on the domain \mathcal{D} .

By the properties of the domain, there exists a radius $r_0 > 0$ and a family of balls $B_{r_0}(q) \subseteq \mathcal{D}$, $q \in \partial\Omega_{r_0}$ such that, for each such q , the closed ball $\bar{B}_{r_0}(q)$ touches $\partial\mathcal{D}$ at exactly one point.

Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' as in (1.7). It follows from Lemma 2.1 (after a simple re-scaling) that there exists an $\lambda_* > 0$, depending only on ϵ , D , W , and \mathcal{D} (in terms of r_0), and a global minimizer $u_{r_0}^q$ of the associated energy to the equation of (3.9) in $W_0^{1,2}(B_{r_0}(q))$ such that

$$0 < u_{r_0}^q(x) < \mu, \quad x \in B_{r_0}(q), \quad \text{and} \quad u_{r_0}^q(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(r_0 - D\lambda^{-1})}(q),$$

provided that $\lambda \geq \lambda_*$. (Without loss of generality, we may assume that $\lambda_* > \lambda_0$). Thanks to Lemma A.2, we obtain that $u_\lambda(x) \geq u_{r_0}^q(x)$, $x \in B_{r_0}(q)$. Since the center q was any point on $\partial\mathcal{D}_{r_0}$, it follows that assertion (4.1) holds true for x such that

$$D\lambda^{-1} \leq \text{dist}(x, \partial\mathcal{D}) \leq 2r_0 - D\lambda^{-1}. \quad (4.2)$$

If $W'(t) < 0$, $t \in [\mu - 2\epsilon, \mu]$, then the validity of (4.1), over the entire specified domain, follows via Lemma A.3 (this is also the case when $W'(t) \leq 0$ for $\mu - t > 0$ small, recall Remark 2.3). Otherwise, we proceed as follows, see also Lemma 2 in [58]: Firstly, we cover $\bar{\mathcal{D}}_{r_0}$ by a finite number of balls of radius $\frac{r_0}{2}$ with centers on $\bar{\mathcal{D}}_{r_0}$. Secondly, if necessary, we increase the value of λ_* such that $D\lambda_*^{-1} < \frac{r_0}{2}$. Lastly, we apply Lemma A.2 to show that

$$u_\lambda(x) \geq u_{r_0}^p(x) \geq \mu - \epsilon, \quad x \in B_{(r_0 - D\lambda^{-1})}(p) \supseteq B_{\frac{r_0}{2}}(p),$$

for every center p of the finite covering of $\bar{\mathcal{D}}_{r_0}$, if $\lambda \geq \lambda_*$. The desired estimate (4.1) now follows from the comments leading to (4.2) and the above relation.

The proof of the proposition is complete. \square

Remark 4.1. A similar result also holds if the domain \mathcal{D} is unbounded.

Remark 4.2. The asymptotic behavior of uniformly bounded from above and below (in λ), stable solutions of (3.9), where $\mathcal{D} \subseteq \mathbb{R}^n$ is bounded and smooth, as $\lambda \rightarrow \infty$, has been studied in [38] in dimensions $n = 2, 3$ by techniques that were used for the proof of De Giorgi's conjecture in low dimensions. In fact, since global minimizers are stable, and since assumption (a') implies that $W'(0) \leq 0$, the assertions of Proposition 4.1 when $n = 2$ follow readily from Theorem 6 in [38]; the same is also true when $n = 3$, provided that the monotonicity assumption (b) from our introduction holds.

Remark 4.3. Let ϵ , D , $R' > 0$ be as in Lemma 2.1. A simple rescaling (see also the proof of Theorem 1.1 in [60]), Lemma 2.1 and Lemma A.2, yield that the solution of (3.9), described in Proposition 4.1, satisfies $\mu - u_\lambda(x) \geq \epsilon$, if $\text{dist}(x, \partial\mathcal{D}) > D\lambda^{-1}$, provided that λ is sufficiently large. Note that relation (2.11) yields the same estimate but over the smaller region $\text{dist}(x, \partial\mathcal{D}) > \frac{R'}{2}\lambda^{-1}$, which depends on n ,

Remark 4.4. Blowing up a global minimizer u_λ of Proposition 4.1 at a point $x_0 \in \partial\mathcal{D}$, up to a subsequence, we find that

$$u_\lambda(x_0 + \lambda y) \rightarrow U(y),$$

uniformly on compacts, as $\lambda \rightarrow \infty$, where U is a nonnegative, global minimizer (in the sense of (2.37)) of the following half-space problem

$$\Delta u = W'(u), \quad y \in \mathbb{R}_+^n; \quad u(y) = 0, \quad y \in \partial\mathbb{R}_+^n,$$

see [12], [38] for more details, where $\mathbb{R}_+^n = \{(y_1, \dots, y_n) : y_1 > 0\}$. This solution is nontrivial by virtue of Remark 4.3. Hence, by the strong maximum principle, recall (a'), we deduce that U is positive in \mathbb{R}_+^n . As before, combining Lemmas 2.1 and A.2, we obtain that

$$u(y) \rightarrow \mu \text{ as } y_1 \rightarrow \infty, \text{ uniformly in } (y_2, \dots, y_n) \in \mathbb{R}^{n-1},$$

(assertion (2.11) is sufficient for this). It follows from Theorem 1.4 in [18] that U depends only on the y_1 variable and is $\mathbf{U}(y_1)$ as described in (1.8). (If $W''(\mu) > 0$ then this has been shown earlier in [12], see also [17], [32] for the weaker case (2.22)).

Remark 4.5. By adapting the proof of Lemma 2.3 in [60] and that of our Proposition 4.1, we can study the boundary layer of globally minimizing solutions of inhomogeneous singular perturbation problems of the form

$$\varepsilon^2 \Delta u = W_u(u, x), \quad x \in \mathcal{D}; \quad u(x) = 0, \quad x \in \partial\mathcal{D},$$

as $\varepsilon \rightarrow 0$, for appropriate righthand side that is more general than the ones considered in [19, 20, 45, 60] (roughly, we want (a') to hold for every fixed x with $a(x)$ instead of μ , for a smooth function a).

APPENDIX A. SOME USEFUL ‘‘COMPARISON’’ LEMMAS OF THE CALCULUS OF VARIATIONS

The following is essentially Lemma 2.1 in [50].

Lemma A.1. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and let $v \in W^{1,2}(\mathcal{O})$. Define $\tilde{v} : \mathcal{O} \rightarrow \mathbb{R}$ as

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } v(x) \in [0, \mu], \\ \mu & \text{if } v(x) \in (-\infty, -\mu) \cup (\mu, \infty), \\ -v(x) & \text{if } v(x) \in (-\mu, 0). \end{cases}$$

Then $\tilde{v} \in W^{1,2}(\mathcal{O})$ and, if W is C^2 and satisfies (a'), we have

$$\int_{\mathcal{O}} \left\{ \frac{1}{2} |\nabla \tilde{v}|^2 + W(\tilde{v}) \right\} dx \leq \int_{\mathcal{O}} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dx.$$

Proof. (Sketch) Firstly, note that $\tilde{v} = G(v)$, $x \in \mathcal{O}$, for some Lipschitz (piecewise linear) function $G : \mathbb{R} \rightarrow \mathbb{R}$. Thus, $\tilde{v} \in W^{1,2}(\mathcal{O})$, see for instance [46]. Then, to finish, note that

$$|\nabla \tilde{v}| \leq |\nabla v| \quad \text{and, thanks to (a'),} \quad W(\tilde{v}) \leq W(v) \quad \text{a.e. in } \mathcal{O},$$

(the former inequality may be proven as in Lemma 2.9 in [73]). □

The following is Lemma 2.3 in [37], which is reproduced in Lemma 1 in [58] and Lemma 2.1 in [60], see also Theorem 1.4 in [49] and Lemma 3.1 in [57].

Lemma A.2. Let \mathcal{D} be a bounded domain in \mathbb{R}^n with smooth boundary. Let $g_1(x, t), g_2(x, t)$ be locally Lipschitz functions with respect to t , measurable functions with respect to x , and for any bounded interval I there exists a constant C such that $\sup_{x \in \mathcal{D}, t \in I} |g_i(x, t)| \leq C$, $i = 1, 2$, holds. Let

$$G_i(x, t) = \int_0^t g_i(x, s) ds, \quad i = 1, 2.$$

For $\eta_i \in W^{1,2}(\mathcal{D})$, $i = 1, 2$, consider the minimization problem:

$$\inf \left\{ J_i(u; \mathcal{D}) \mid u - \eta_i \in W_0^{1,2}(\mathcal{D}) \right\}, \quad \text{where} \quad J_i(u; \mathcal{D}) = \int_{\mathcal{D}} \left\{ \frac{1}{2} |\nabla u|^2 - G_i(x, u) \right\} dx.$$

Let $u_i \in W^{1,2}(\mathcal{D})$, $i = 1, 2$, be minimizers to the minimization problems above. Assume that there exist constants $m < M$ such that

- $m \leq u_i(x) \leq M$ a.e. for $i = 1, 2$, $x \in \mathcal{D}$,
- $g_1(x, t) \geq g_2(x, t)$ a.e. for $x \in \mathcal{D}$, $t \in [m, M]$,
- $M \geq \eta_1(x) \geq \eta_2(x) \geq m$ a.e. for $x \in \mathcal{D}$.

Suppose further that $\eta_i \in W^{2,p}(\mathcal{D})$ for any $p > 1$, and that they are *not identically equal* on $\partial\mathcal{D}$. Then, we have

$$u_1(x) \geq u_2(x), \quad x \in \mathcal{D}.$$

The following is a nontrivial extension of Lemma A.1, see [7].

Lemma A.3. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded connected set with Lipschitz boundary and $W : \mathbb{R}^m \rightarrow \mathbb{R}$, $m \geq 1$, be a C^2 nonnegative potential such that

$$\lambda \rightarrow W(Q + \lambda e), \quad \text{with } |e| = 1,$$

forms a strictly increasing function on $[0, r_0)$, where $Q \in \mathbb{R}^m$ is a global minimum of W such that $W(Q) = 0$ and r_0 is a fixed positive constant. Further, let $\mathcal{A} \subset \Omega$ be an open set with nonempty Lipschitz boundary. Moreover, assume that

- (1) $u \in W^{1,2}(\Omega) \cap C^1(\Omega)$,
- (2) $|u(x) - Q| \leq r$ on $\partial\mathcal{A} \cap \Omega$ for some $r \in (0, \frac{r_0}{2})$,
- (3) $|u(x_0) - Q| > r$ for some $x_0 \in \mathcal{A}$.

Then, there exists $\tilde{u} \in W^{1,2}(\Omega)$ such that

$$\begin{cases} \tilde{u}(x) = u(x), & x \in \Omega \setminus \mathcal{A}, \\ |\tilde{u}(x) - Q| \leq r, & x \in \mathcal{A}, \\ \int_{\Omega} \left\{ \frac{1}{2} |\nabla \tilde{u}|^2 + W(\tilde{u}) \right\} dx < \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx. \end{cases}$$

For an extension of the above lemma, when $n = 1$ and the ball $B_r(Q) \subset \mathbb{R}^m$ is replaced by a convex set, see [8].

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REFERENCES

- [1] A. AFTALION, S. ALAMA, and L. BRONSARD, *Giant vortex and the breakdown of strong pinning in a rotating Bose–Einstein condensate*, Arch. Ration. Mech. Anal. **178** (2005), 247–286.
- [2] N. D. ALIKAKOS, and H. C. SIMPSON, *A variational approach for a class of singular perturbation problems and applications*, Proc. Roy. Soc. Edinburgh Sect. A **107** (1987), 27–42.
- [3] N. D. ALIKAKOS, and P. W. BATES, *On the singular limit in a phase field model of phase transitions*, Ann. Inst. Henri Poincaré **5** (1988), 141–178.
- [4] N. D. ALIKAKOS, *Some basic facts on the system $\Delta u - W_u(u) = 0$* , Proc. Amer. Math. Soc. **139** (2011), 153–162.
- [5] N. D. ALIKAKOS, and G. FUSCO, *Entire solutions to equivariant elliptic systems with variational structure*, Arch. Rat. Mech. Anal. **202** (2011), 567–597.
- [6] N. D. ALIKAKOS, *A new proof for the existence of an equivariant entire solution connecting the minima of the potential for the system $\Delta u - W_u(u) = 0$* , Arxiv preprint (2011).
- [7] N. D. ALIKAKOS, and G. FUSCO, *A replacement lemma for obtaining pointwise estimates in phase transition models*, Arxiv preprint (2011).
- [8] N. D. ALIKAKOS, and N. KATZOURAKIS, *Heteroclinic travelling waves of gradient diffusion systems*, Trans. Amer. Math. Soc. **363** (2011), 1362–1397.
- [9] N. D. ALIKAKOS, and P. SMYRNELIS, *Existence of lattice solutions to semilinear elliptic systems with periodic potential*, Electr. J. Diff. Equations **2012** (2012), pp. 1–15.
- [10] N. D. ALIKAKOS, and A. FALIAGAS, *The stress–energy tensor and Pohozaev’s identity for systems*, Acta Math. Scientia **32** (2012), 433–439.
- [11] H. AMANN, *A uniqueness theorem for nonlinear elliptic boundary value problems*, Arch. Rational Mech. Anal. **44** (1971/72), 178–181.
- [12] S. B. ANGENENT, *Uniqueness of the solution of a semilinear boundary value problem*, Math. Annalen **272** (1985), 129–138.
- [13] V. I. ARNOL’D, Ordinary Differential Equations, Springer-Verlag, 1992.
- [14] M. BARDI, and B. PERTHAME, *Exponential decay to stable states in phase transitions via a double Log-transformation*, Comm. Partial Diff. Eqns. **15** (1990), 1649–1669.
- [15] H. BERESTYCKI, and P. L. LIONS, *Some applications of the method of sub- and supersolutions*, Lecture Notes in Math. **782** (1980), Springer, Berlin, 16–41.
- [16] H. BERESTYCKI, and P. L. LIONS, *Nonlinear scalar field equations I: Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313–347.
- [17] H. BERESTYCKI, F. HAMEL, and R. MONNEAU, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. Volume **103** (2000), 375–396.
- [18] H. BERESTYCKI, L. CAFFARELLI, and L. NIRENBERG, *Further qualitative properties for elliptic equations in unbounded domains*, Ann Scuola Norm Sup Pisa **25** (1997), 69–94.
- [19] M. S. BERGER, and L. E. FRAENKEL, *On the asymptotic solution of a nonlinear Dirichlet problem*, J. Math. Mech. **19** (1970), 553–585.
- [20] M. S. BERGER, and L. E. FRAENKEL, *On singular perturbations of nonlinear operator equations*, Indiana Univ. Math. J. **20** (1971), 623–31.
- [21] F. BÉTHUEL, H. BREZIS, and F. HÉLEIN, *Asymptotics for the minimization of a Ginzburg–Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), 123–148.
- [22] F. BETHUEL, G. ORLANDI, and D. SMETS, *Slow motion for gradient systems with equal depth multiple-well potentials*, J. Differential Equations **250** (2011), 53–94.
- [23] H. BREZIS, and L. OSWALD, *Remarks on sublinear elliptic equations*, Nonlin. Anal. **10** (1986), 55–64.
- [24] H. BREZIS, *Symmetry in Nonlinear PDE’s*, in Proc. Symp. Pure Math. **65**, Florence, 1996, Amer. Math. Soc., 1999, 1–12.
- [25] F. BROCK, *Rearrangements and Applications to Symmetry Problems in PDE*, in M. Chipot, editor, Handbook of Differential Equations: Stationary Partial Differential Equations, vol. **1**, pages 1–61. Elsevier, 2007.
- [26] K. J. BROWN, and H. BUDIN, *On the existence of positive solutions for a class of semilinear elliptic boundary value problems*, SIAM J. Math. Anal. **10** (1979), 875–883.

- [27] X. CABRÉ, and A. CAPELLA, *On the stability of radial solutions of semilinear elliptic equations in all of \mathbb{R}^n* , C. R. Acad. Sci. Paris, Ser. I **338** (2004), 769–774.
- [28] X. CABRÉ, and J. TERRA, *Saddle-shaped solutions of bistable diffusion equations in all of \mathbb{R}^{2m}* , J. Eur. Math. Soc. **11** (2009), 819–843.
- [29] N. CHAFEE, and E. F. INFANTE, *A bifurcation problem for a nonlinear partial differential equation of parabolic type*, Applicable Anal. **4** (1974/75), 17–37.
- [30] X. CHEN, *Global asymptotic limit of solutions of the Cahn–Hilliard equation*, J. Diff. Geom. **44** (1996), 262–311.
- [31] P. CLÉMENT, and L. A. PELETIER, *On a nonlinear eigenvalue problem occurring in population genetics*, Proc. Royal Soc. Edinburgh **100A**, 85–101 (1985).
- [32] P. CLÉMENT, and G. SWEERS, *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa **14** (1987), 97–121.
- [33] E. A. CODDINGTON, and N. LEVINSON, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.
- [34] E. N. DANCER, *On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large*, Proc. London Math. Soc. **53** (1986), 429–452.
- [35] E. N. DANCER, and J. WEI, *On the profile of solutions with two sharp layers to a singularly perturbed semilinear Dirichlet problem*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 691–701.
- [36] E. N. DANCER, and J. WEI, *On the location of spikes of solutions with two sharp layers for a singularly perturbed semilinear Dirichlet problem*, J. Differential Equations **157** (1999), 82–101.
- [37] E. N. DANCER, and S. YAN, *Construction of various types of solutions for an elliptic problem*, Calc. Var. Partial Differential Equations **20** (2004), 93–118.
- [38] E. N. DANCER, *Stable and finite Morse index solutions on \mathbb{R}^n or on bounded domains with small diffusion*, Trans. Amer. Math. Soc. **357** (2005), 1225–1243.
- [39] E. N. DANCER, *Stable and finite Morse index solutions on \mathbb{R}^n or on bounded domains with small diffusion II*, Indiana Univ. Math. J. **53** (2004), 97–108.
- [40] H. DANG, P. C. FIFE, and L. A. PELETIER, *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. **43** (1992), 984–998.
- [41] C. DE COSTER, and S. NICAISE, *Lower and upper solutions for elliptic problems in nonsmooth domains*, J. Differential Equations **244** (2008), 599–629.
- [42] D. G. DE FIGUEIREDO, *On the existence of multiple ordered solutions of nonlinear eigenvalue problems*, Nonlinear Anal. **11** (1987), 481–492.
- [43] M. DEL PINO, M. KOWALCZYK, and J. WEI, *On De Giorgi conjecture in dimensions $N \geq 9$* , Annals of Mathematics **174** (2011), 1485–1569.
- [44] M. DEL PINO, M. MUSSO, and F. PACARD, *Solutions of the Allen-Cahn equation which are invariant under screw motion*, Manuscripta Mathematica **138** (2012), 273–286.
- [45] J. M. DE VILLIERS, *A uniform asymptotic expansion of the positive solution of a non linear Dirichlet problem*, Proc. London Math. Soc. **27** (1973), 701–722.
- [46] L. C. EVANS, and R. F. GARIEPY, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.
- [47] P. C. FIFE, *Semilinear elliptic boundary value problems with small parameters*, Arch. Rational Mech. Anal. **52** (1973), 205–232.
- [48] P. C. FIFE, H. KIELHÖFER, S. MAIER-PAAPE, and T. WANNER, *Perturbation of doubly periodic solution branches with applications to the Cahn–Hilliard equation*, Physica D **100** (1997), 257–278.
- [49] A. FRIEDMAN, and D. PHILLIPS, *The free boundary of a semilinear elliptic equation*, Trans. Amer. Math. Soc. **282** (1984), 153–182.
- [50] G. FUSCO, F. LEONETTI, and C. PIGNOTTI, *A uniform estimate for positive solutions of semilinear elliptic equations*, Trans. Amer. Math. Soc. **363** (2011), 4285–4307.
- [51] B. GIDAS, W. M. NI, and L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
- [52] D. GILBARG, and N. S. TRUDINGER, *Elliptic partial differential equations of second order*, second ed., Springer-Verlag, New York, 1983.
- [53] P. GRISVARD, *Elliptic problems in nonsmooth domains*, Classics in applied mathematics, SIAM, 2011

- [54] P. HESS, *On multiple solutions of nonlinear elliptic eigenvalue problems*, Comm. Partial Diff. Eqns. **6** (1981), 951–961.
- [55] L. HÖRMANDER, *The analysis of linear partial differential operators III: Pseudo-differential operators*, Springer-Verlag, 1985.
- [56] J. JANG, *On spike solutions of singularly perturbed semilinear Dirichlet problems*, J. Differential Equations **114** (1994), 370–395.
- [57] D. JERISON, and R. MONNEAU, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Annali di Matematica **183** (2004), 439–467.
- [58] K. KURATA, and H. MATSUZAWA, *Multiple stable patterns in a balanced bistable equation with heterogeneous environments*, Applicable Analysis **89** (2010), 1023–1035.
- [59] L. LASSOUED, and P. MIRONESCU, *Ginzburg-Landau type energy with discontinuous constraint*, J. Anal. Math. **77** (1999), 1–26.
- [60] G. LI, J. YANG, and S. YAN, *Solutions with boundary layer and positive peak for an elliptic Dirichlet problem*, Proc. Royal Soc. Edinburgh **134A** (2004), 515–536.
- [61] F.-H. LIN, *Static and moving vortices in Ginzburg-Landau theories*, in Nonlinear partial differential equations in geometry and physics (Knoxville, TN, 1995), 71–111, Progr. Nonlinear Differential Equations Appl. **29**, Birkhäuser, Basel, 1997.
- [62] P. L. LIONS, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review **24** (1982), 441–467.
- [63] O. LOPES, *Radial and nonradial minimizers for some radially symmetric functionals*, Electr. J. Diff. Equations **1996** (1996), 1–14.
- [64] H. MATANO, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci. **15** (1979), 401–454.
- [65] W. -M. NI, *On the elliptic equation $\Delta U + KU^{(n+2)/(n-2)} = 0$, its generalization and application in geometry*, Indiana J. Math. **4** (1982), 493–529.
- [66] W.-M. NI, *Qualitative properties of solutions to elliptic problems*, in M. Chipot and P. Quittner, editors, Handbook of Differential Equations: Stationary Partial Differential Equations, vol. **1**, pages 157–233. Elsevier, 2004.
- [67] E. S. NOUSSAIR, *On semilinear elliptic boundary value problems in unbounded domains*, J. Differential Equations **41** (1981), 334–348.
- [68] E. S. NOUSSAIR, and C. A. SWANSON, *Global positive solutions of semilinear elliptic problems*, Pacific J. Math. **115** (1984), 177–192.
- [69] A. OGATA, *On existence and multiplicity theorems for semilinear elliptic equations in exterior domains*, Funkcial. Ekvac. **27** (1984), 281–299.
- [70] F. PACARD, *Geometric aspects of the Allen-Cahn equation*, Matematica Contemporânea **37** (2009), 91–122.
- [71] P. POLACIK, *On symmetry of nonnegative solutions of elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 1–19.
- [72] P. H. RABINOWITZ, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973/74), 173–186.
- [73] J. ROBINSON, *Infinite-dimensional dynamical systems*, Cambridge University Press, 2001.
- [74] D. H. SATTINGER, *Topics in stability and bifurcation theory*, Lecture Notes in Math. **309**, Springer, Berlin, 1973.
- [75] J. SERRIN, *A remark on the preceding paper of Amann*, Arch. Rational Mech. Anal. **44** (1971/72), 182–186.
- [76] J. SHI, *Solution set of semilinear elliptic equations: Global bifurcation and exact multiplicity*, World Scientific Publ., 2008.
- [77] I. M. SIGAL, *Applied Partial Differential Equations MAT1508/APM446*, Lecture notes that are available on the author’s webpage, 2012.
- [78] J. SMOLLER, and A. WASSERMAN, *Existence of positive solutions for semilinear elliptic equations in general domains*, Arch. Ration. Mech. Anal. **98** (1987), 229–249.
- [79] G. SWEERS, *On the maximum of solutions for a semilinear elliptic problem*, Proc Royal Soc Edinburgh A **108** (1988), 357–370.

- [80] A. TERTIKAS, *Stability and instability of positive solutions of semiposition problems*, Proc. Amer. Math. Soc. **114** (1992), 1035–1040.
- [81] S. VILLEGAS, *Nonexistence of nonconstant global minimizers with limit at ∞ of semilinear elliptic equations in all of \mathbb{R}^n* , Comm. Pure Appl. Anal. **10** (2011), 1817–1821.
- [82] W. WALTER, *Ordinary differential equations*, Graduate texts in mathematics **182**, Springer-Verlag, New York, 1998.
- [83] J. WEI, *Exact multiplicity for some nonlinear elliptic equations in balls*, Proc. Amer. Math. Soc. **125** (1997), 3235–3242.

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